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# METHOD OF CONSTRUCTING AND INVESTIGATING STABILITY OF PERIODIC MOTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS

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An algorithm is proposed for the construction and investigation of orbital stability of Liapunov motions close to the equilibrium position of autonomous Hamiltonian systems with many degrees of freedom.

**1.** Let us consider the autonomous system of Hamilton's differential equations with $N+1$ degrees of freedom. We select the generalized coordinates and moments so that the origin of phase space coordinates is the equilibrium position, and represent the Hamiltonian, which is analytic in the neighborhood of the equilibrium position, in the form of series

$$H = H_2 + \ldots + H_m + \ldots \tag{1.1}$$

where $H_m$ is a homogeneous polynomial of power $m$ in coordinates and momenta whose coefficients depend on the problem parameters.

First, we consider the system with Hamiltonian $H_2$, representing the determining equation in the form

$$D(\sigma) = |a_{jk} - \sigma\delta_{jk}| = 0 \tag{1.2}$$

where $\delta_{jk}$ $(j,\ k = 0,\ 1,\ \ldots\ N)$ is the Kronecker delta and $a_{jk}$ are coefficients of the linear system of differential equations.

If Eq. (1.2) has a pair of pure imaginary roots of the form $\sigma_0 = \pm i\lambda_0$ and among its other roots there are none equal to $\sigma_0$, then the coordinates and momenta $q_0, q_j, p_0,$ and $p_j$ $(j = 1, \ldots, N)$ can be chosen so that function $H_2$ is of the form

$$H_2 = {}^1/_2\, \lambda_0 (q_0^2 + p_0^2) + H_2^{(N)} \tag{1.3}$$

where $H_2^{(N)}$ is a function that depends only on the variables $q_j$ and $p_j$. It can be checked that the differential equations which correspond to the Hamiltonian (1.3) have the particular solution

$$q_0 = \alpha_0 \sin(\lambda_0 t + \beta_0), \qquad p_0 = \alpha_0 \cos(\lambda_0 t + \beta_0) \tag{1.4}$$
$$q_j = p_j = 0 \qquad (j = 1, \ldots, N)$$

where $\alpha_0$ and $\beta_0$ are arbitrary constants that depend on initial conditions. Solution (1.4) is periodic of period $T_0 = 2\pi / |\lambda_0|$.

If Eq. (1.2) has no other roots of the form $\pm in_0\lambda_0$ ($n_0$ is an integer), then according to Liapunov's theorem on the holomorphic integral [1], a periodic motion of period close to $T_0$ exists in the nonlinear system of differential equations with the Hamiltonian $H$. The periodic motions constitute a set whose parameter is called the amplitude (or the orbital parameter) of periodic motion $\varepsilon$ and depends in initial conditions. Liapunov had also proposed a procedure for constructing periodic motions in the form of series in powers of the orbital parameter $\varepsilon$. Since then this problem was extensively developed, and at present many methods of constructing periodic motions are available. A method of constructing periodic motions based on canonical transformations is also proposed here, although the main aim is to develop an algorithm for solving the problem of periodic motion stability in the strictly nonlinear meaning. The derived method of periodic motion construction is adapted to the solution of this basic problem. This paper is a continuation and natural generalization of the earlier investigation by the authors (*).

**2.** Periodic motions are Liapunov unstable relative to perturbations of coordinates and momenta $q_0$ and $p_0$, since their period depends on initial conditions. The problem of periodic motion orbital stability is, however, interesting.

In that method the value of constant energy is not fixed, and it can be varied in a certain range. Hence no reduction is made to the number of degrees of freedom, as used in the isoenergetic reduction. This makes it possible to investigate the complete neighborhood of periodic motion using canonical transformations, and in the periodic motion neighborhood it is possible to introduce such local coordinates that the Hamiltonian of perturbed motion is of a form similar to the normal form in the neighborhood of the equilibrium position. In this way the orbital stability problem of periodic motion reduces to that of Liapunov stability relative to local coordinates.

In the considered problem the constructive application of the local method may be schematically outlined as the sequence of the following operations:

1) determination of the investigated periodic motion in "action-angle" variables;

2) introduction of local coordinates in the periodic motion neighborhood, and the determination of the perturbed motion Hamiltonian;

3) passing to the new "angle" variable, linear normalization, and obtaining statements about stability in linear approximation;

4) reverting to the old independent variable and effecting the nonlinear normalization of the Hamiltonian;

5) using the properties of coefficients of the Hamiltonian normal form to determine the orbital stability of the periodic motion.

**3.** When among the roots of the determining equation (1.2) there is at least one pair of roots with nonzero real part, the considered periodic motion is unstable. In what follows we assume that all roots of Eq. (1.2) are pure imaginary and that there are among them no equal or zero roots. It can be then considered that the quadratic part of the Hamiltonian (1.1) is of the form

$$H_2 = \frac{1}{2}\lambda_0(q_0^2 + p_0^2) + \frac{1}{2}\sum_{j=1}^{N}\lambda_j(q_j^2 + p_j^2) \tag{3.1}$$

where $|\lambda_j|$ are frequencies of the linear system with the Hamiltonian.

*) Markeev, A. P. and Sokol'skii, A. G., Investigation of periodic motions close to Lagrange solutions of the limited three-body problem. Preprint No. 110, Inst. Prikl. Matem. Akad. Nauk SSSR, 1975.

The form of $H_m$ in (1.1) is

$$H_m = \sum_{\nu_0+\ldots+\mu_N=m} h_{\nu_0\mu_0\nu_1\mu_1\ldots\nu_N\mu_N} q_0^{\nu_0} p_0^{\mu_0} q_1^{\nu_1} p_1^{\mu_1} \ldots q_N^{\nu_N} p_N^{\mu_N} \qquad (3.2)$$

For the construction of periodic motions of the nonlinear problem we use the method of canonical transformations, and represent forms (3.2) as

$$H_m = \sum_{\alpha=0}^{m} H_{\alpha,\beta} \quad (\beta = m - \alpha) \qquad (3.3)$$

where $H_{\alpha,\beta}$ denotes the totality of term of power $\alpha$ of variables $q_0$ and $p_0$ and of power $\beta$ of remaining variables. We now carry out the nonlinear canonical transformation

$$q_i, p_i \to q_i^*, p_i^* \quad (i = 0, 1, \ldots, N), \quad (H \to H^*, H_2 = H_2^*) \qquad (3.4)$$

which in all forms (3.3) of the new Hamiltonian $H^*$ would normalize terms $H^*_{m,0}$ and cancel terms $H^*_{m-1,1}$. Such transformation is convergent [3,4].

Transformation (3.4) can, for instance, be obtained using the classic Birkhoff's method [5]. However that method has a number of disadvantages that are particularly evident in the normalization of the Hamiltonian of systems with many degrees of freedom of considerable order relative to coordinates and momenta. Because of this transformation (3.4) and all subsequent normalizing transformations are better effected by the recently developed method of Hori — Deprit which is based on canonical transformations of Lie [6,7]. It should be noted in this connection that an essential part of the proposed method of periodic motion analysis is the use of the described algorithms on a computer. The basic part of the necessary programs of normalization computation was published by the authors earlier (*).

We represent the generating function of transformation (3.4) which depends only on new (or only on old) variables in the form of series

$$T = T_3 + \ldots + T_m + \ldots$$

The operator equation for the determination of coefficients of the generating function and of the new Hamiltonian is of the form

$$D_0 T_m = H_m' - H_m^* \quad (m = 3, 4, \ldots) \qquad (3.5)$$

where operator $D_0$ is defined as follows:

$$D_0 T_m = -\{H_2; T_m\} = -\sum_{i=0}^{N} \left[\frac{\partial H_2}{\partial q_i}\frac{\partial T_m}{\partial p_i} - \frac{\partial H_2}{\partial p_i}\frac{\partial T_m}{\partial q_i}\right]$$

Here and in what follows braces denote the operation of computation of Poisson's braces. The forms of $H_m'$ in (3.5) are defined in terms of "lower" functions

$$H_\alpha, H_\beta^*, T_\beta \ (\alpha = 2, \ldots, m; \ \beta = 3, \ldots, m-1) \qquad (3.6)$$

*) Markeev, A. P. and Sokol'skii, A. G., Certain computational algorithms for normalizing Hamiltonian systems. Preprint No. 31, Inst. Prikl. Matem. Akad. Nauk SSSR, 1976.

using Poisson's braces $\{F_\alpha;\ T_\beta\} = D_\beta F_\alpha$, where functions $F_\alpha$ are expressed in turn in terms of functions (3.6) using similar Poisson's braces. For example,

$$H_3' = H_3, \quad H_4' = H_4 + {}^1/_2\, D_3\,(H_3 + H_3^*) \tag{3.7}$$
$$H_5' = H_5 + {}^1/_2\, D_3\,[H_4 + H_4^* + {}^1/_6 D_3\,(H_3 - H_3^*)] +$$
$${}^1/_2 D_4\,(H_3 + H_3^*)$$

If in the course of solving a specific problem it is necessary to consider in the Hamiltonian $H$ terms $H_m$ of order higher than the fifth, then $H_m'$ $(m \geqslant 6)$ may be calculated by the formulas proposed by the authors (see footnote on p. 53).

Equation (3.5) decomposes into groups that correspond to terms $H_{\alpha,\beta}$ in formula (3.3). Hence such terms may be normalized independently of each other. Denominators of the form

$$d_{\alpha,\beta} = |\lambda_0|(\nu_0 - \mu_0) + \sum_{j=1}^{N} |\lambda_j|(\nu_j - \mu_j) \tag{3.8}$$
$$\nu_0 + \mu_0 = \alpha, \qquad \sum_{j=1}^{N} (\nu_j + \mu_j) = \beta$$

appear in the normalization of terms $H_{\alpha,\beta}$ in the form (3.2).

Assume that the Liapunov condition of existence of periodic motions of period $2\pi/|\lambda_0|$ is satisfied, i.e. that the problem parameters are such that for all integral $n_0$ the inequalities

$$\lambda_j \neq n_0\lambda_0 \quad (j = 1, \ldots N) \tag{3.9}$$

are satisfied.

Then taking into account (3.9) and that parameters $\lambda_i$ $(i = 0, 1, \ldots, N)$ are nonzero, we find that in all forms of $H_m^*$ in the new Hamiltonian terms of form $H_{m-1,1}^*$ can be completely cancelled, since denominators $d_{\alpha,\beta}$ which correspond to these terms do not vanish. This also implies that in forms of odd power (i.e. in $H_{2m-1}^*$) the terms $H_{2m-1,0}^*$ can be cancelled.
In forms of even powers these terms can be normalized and represented in the form

$$H_{2m,0}^* = b_{2m} 2^{-m} (q_0^{*2} + p_0^{*2})^m \tag{3.10}$$

where $b_{2m}$ $(2m = 4, 6, \ldots)$ depend only on parameters of the problem.

When Eqs. (3.5) have been solved for all $m$ and the related terms of expansion of the generating function determined, the obtained transformation (3.4) is of the form

$$q_i = q_i^* + \sum_{k=1}^{\infty} \frac{1}{k!} D^k q_i^*, \quad p_i = p_i^* + \sum_{k=1}^{\infty} \frac{1}{k!} D^k p_i^* \quad (i = 0, 1, \ldots, N) \tag{3.11}$$
$$D = \sum_{m=3}^{\infty} D_m, \quad D^0 f \equiv f, \quad D^1 f \equiv Df = \{f;\ T\}, \ldots,$$
$$D^{k+1} f = D\,(D^k f), \ldots$$

where $D$ is the differential Lie operator and $f$ is an arbitrary function of variables $q_i^*$ and $p_i^*$ (or $q_i$ and $p_i$ ).

Let transformation (3.4) have been carried out. Then, retaining previous notation (without asterisks) for the coordinates, momenta and the Hamiltonian, we find that in the new Hamiltonian $H$ the totality of variables $q_j$ and $p_j$ $(j = 1, \ldots, N)$ is of a power not lower than the second. This means that the equations of motion admit particular solutions that correspond to Liapunov's periodic motions for which $q_j = p_j = 0$ and the variation of $q_0$ and $p_0$ is defined by the equations

$$\frac{dq_0}{dt} = \frac{\partial H}{\partial p_0} = \left[\lambda_0 + \sum_{m=2}^{\infty} m2^{1-m} b_{2m} (q_0^2 + p_0^2)^{m-1}\right] p_0 \tag{3.12}$$

$$\frac{dp_0}{dt} = -\frac{\partial H}{\partial q_0} = -\left[\lambda_0 + \sum_{m=2}^{\infty} m2^{1-m} b_{2m} (q_0^2 + p_0^2)^{m-1}\right] q_0$$

In variables action $I$ -angle $W$, related to $q_0$ and $p_0$ by the indicated below formulas Eqs. (3.12), are of the form

$$\frac{dI}{dt} = 0, \quad \frac{dW}{dt} = \lambda_0 + \sum_{m=2}^{\infty} mb_{2m} I^{m-1} \tag{3.13}$$

$$q_0 = \sqrt{2I} \sin W, \quad p_0 = \sqrt{2I} \cos W$$

The solution of these equations is

$$I = I_0 = \text{const}, \quad W = \Omega_0 (I_0)(t - t_0) + W_0 \tag{3.14}$$

$$\Omega_0 = \lambda_0 + \sum_{m=2}^{\infty} mb_{2m} I_0^{m-1}, \quad T_0 = \frac{2\pi}{|\Omega_0|}$$

where $\Omega_0$ is the frequency and $T_0$ the period of periodic motion.

It will be seen that when $I_0 \to 0$ the motion period tends to $2\pi / |\lambda_0|$.

**4.** Let us investigate the stability of periodic motion (3.14) relative to perturbations of periodic motion frequency (or what is the same, relative to the perturbation of action variable $I_0$ of unperturbed periodic motion) and to perturbations of $q_j$ and $p_j$, $(j = 1, \ldots, N)$. Let $\varepsilon = \sqrt{2I_0}$ be a small but finite quantity (only small periodic motions are considered). Let $I$ be the variable of the perturbed motion action related to $I_0$ by the formula

$$I = {}^1/_2\varepsilon^2 + r_0 \tag{4.1}$$

where $r_0$ is the perturbation of the variable action. The sign of quantity $r_0$ is arbitrary, and $q_j$ and $p_j$ are quantities of the first and $r_0$ of the second order of smallness, and in their meaning all these quantities, unlike $\varepsilon$, are infinitely small.

Using (4.1) we define the Cartesian coordinates $q_0$ and $p_0$ of perturbed motion in terms of $r_0$ and $\varepsilon$ as follows:

$$q_0 = \sqrt{2I} \sin W, \quad p_0 = \sqrt{2I} \cos W \tag{4.2}$$

$$\sqrt{2I} = \varepsilon\left\{1 + \frac{r_0}{\varepsilon^2} + \sum_{m=1}^{\infty}(-1)^m \frac{(2m-1)!!}{(m+1)!}\left(\frac{r_0}{\varepsilon^2}\right)^{m+1}\right\} =$$
$$\varepsilon + \frac{r_0}{\varepsilon} - \frac{r_0^2}{2\varepsilon^3} + O(r_0^3)$$

Frequency of the considered periodic motion (3.14) in terms of $\varepsilon$ is of the form

$$\Omega_0 = \sum_{m=0}^{\infty} \Omega_0^{(m)}\varepsilon^m = \lambda_0 + c_{00}\varepsilon^2 + O(\varepsilon^4) \tag{4.3}$$
$$\Omega_0^{(0)} = \lambda_0, \quad \Omega_0^{(2m-1)} = 0, \quad \Omega_0^{(2m)} = (m+1)2^{-m} b_{2m+2}$$

where $c_{00}$ denote the quantity $b_4$ from (3.10).

Substituting into the Hamiltonian the quantities (4.2) and collecting terms of like order relative to $q_j, p_j$ and $\sqrt{|r_0|}$ for the Hamiltonian of perturbed motion we obtain

$$K = K_2 + K_3 + K_4 + \ldots \tag{4.4}$$
$$K_2 = \Omega_0 r_0 + \frac{1}{2}\sum_{j=1}^{N}\lambda_j(q_j^2 + p_j^2) + \sum_{m=1}^{\infty}\hat{H}_{m,2}, \quad K_3 = \sum_{m=0}^{\infty}\hat{H}_{m,3}$$
$$K_4 = B_{00}r_0^2 + \left[\frac{1}{\varepsilon^2}\sum_{m=1}^{\infty}\hat{\overline{H}}_{m,2}\right]r_0 + \sum_{m=0}^{\infty}\hat{H}_{m,4}$$
$$\overline{H}_{m,2} = \left(q_0\frac{\partial}{\partial q_0} + p_0\frac{\partial}{\partial p_0}\right)H_{m,2}$$
$$B_{00} = c_{00} + \sum_{m=1}^{\infty} B_{00}^{(2m)}\varepsilon^{2m}, \quad B_{00}^{(2m)} = (m+1)(m+2)2^{-(m+1)}b_{2m+4}$$

where the superscript $\wedge$ indicates that the expressions $\varepsilon \sin W$ and $\varepsilon \cos W$ are to be substituted for $q_0$ and $p_0$ in the corresponding forms.

Hamiltonian (4.4) is of the period $2\pi$ relative to the variable $W$. Dots in (4.4) denote terms whose order of smallness relative to perturbations is not lower than the sixth. The Hamiltonian of perturbed motion depends first of all on the problem input parameters $\mathbf{U}$ and, secondly on parameter $\varepsilon$ which defines the periodic motion amplitude (3.14). Note that in the problem of stability the dependence on the initial instant of time $t_0$ is unimportant.

The investigation of stability in a specific mechanical system shows that in the normalization process the values of parameters $\mathbf{U}$ for which resonance of the first (Liapunov's condition of periodic motion existence), second (generating point of parametric resonance regions), of the third and fourth orders (generating points for the related resonance surfaces) are possible. The general form of such resonance representation is

$$\sum_{j=1}^{N} n_j\lambda_j = n_0\lambda_0 \quad \left(\sum_{j=1}^{N}|n_j| = n\right) \tag{4.5}$$

where $n$ is the order of resonance and $n_0$ is an arbitrary integer.

When deciding which of resonances are to be taken into account for a complete investigation of stability in multidimensional Hamiltonian systems in the case of a specific problem the following two remarks must be taken into consideration:

a) the dependence of the frequency of a system linearized in the equilibrium position neighborhood on parameters U is known, hence in that region of parameter variation the latter are subjected to certain constraints and, consequently, from among all resonances (4.5) only those that are theoretically possible are to be selected;

b) the structure of the Hamiltonian is often such that some of the theoretically possible resonance do not appear in the course of normalization (i.e. they do not result in the appearance of zero denominators in the generating function), which makes their consideration pointless.

These two remarks considerably lighten the investigation of cases resonance in a specific problem.

**5.** Let us investigate the stability of a linear system with Hamiltonian $K_2$ ( see (4.4) ).

First of all note that the values of parameters U which satisfy relationships (4.5), where $n=2$ and $n_0$ is an arbitrary nonzero integer, are generating parameters for the instability region (region of parametric resonance) in the space of parameters U and $\varepsilon$. Without loss of generality such resonances can be represented in the form

$$n_1\lambda_1 + n_2\lambda_2 = n_0\lambda_0 \quad (n_1 \geqslant 0) \tag{5.1}$$

where $n_1 + |n_2| = 2$. For $n_1 = n_2 = 1$ resonances (5.1) are called parametric resonances of the combinative type, while for $n_1 = 2$ and $n_2 = 0$ ( or $n_1 = 0$ and $n_2 = 2$ ) they are of the basic type. Resonances (5.1) for which $n_1 n_2 < 0$ do not generate instability regions [8]. We shall, therefore, assume that the numbers $n_1$ and $n_2$ are positive.

We shall describe the procedure of normalization of the quadratic part of the perturbed motion Hamiltonian, and represent function $K_2$ as

$$K_2 = \Omega_0 [r_0 + G_2 (q_j, p_j, W)] \tag{5.2}$$

$$G_2 = \sum_{m=0}^{\infty} G_{m,2}\varepsilon^m, \quad G_{0,2} = \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda_0}(q_j^2 + p_j^2)$$

$$G_{m,2} = \frac{1}{\lambda_0}F_{m,2} - \frac{1}{\lambda_0}\sum_{k=1}^{m} G_{m-k,2}\Omega_0^{(k)}, \quad F_{m,2} = \hat{H}_{m,2}\varepsilon^{-m} \quad (m = 1, 2, \ldots)$$

Functions $G_{m,2}$ (and also $F_{m,2}$ ) are of zero order relative to $\varepsilon$, are $2\pi$-periodic functions of $W$, and are expressed by finite series of sines and cosines of integral multiplicities of $W$, and their maximum multiplicity does not exceed $m$.

To reduce function (5.2) to the normal form it is necessary first to normalize it with respect to variables $q_j$ and $p_j$ $(j = 1, \ldots, N)$. For this we pass to the new independent variable $W$. Then the Hamiltonian that defines the variation of variables $q_j$ and $p_j$ is represented by function $G_2$ which corresponds to the nonautonomous canonical system with $N$ degrees of freedom.

Normalization of the Hamiltonian $G_2$ can be carried out by conventional methods, for example, using an algorithm similar to those of Birkhoff or Deprit—Hori. At each step of $G_{m,2}$ normalization it is necessary to solve systems of linear differential equations with periodic coefficients. However in the considered case these equations have a small parameter $\varepsilon$ and the "time" $W$ in the right-hand sides of differential equations appears in a special form. Hence the normalization of the nonautonomous canonical system with Hamiltonian $G_2$ can be reduced to the normalization of an autonomous system (but with $N+1$ degrees of freedom ), i. e. to the solution of a system of algebraic equations.

First of all note that for normalizing function $G_2$ we use the operator equation (3.5) in the form

$$\Delta_0 T_{m,2} = G'_{m,2} - G^*_{m,2} \tag{5.3}$$

$$\Delta_0 = D_0 - \frac{\partial}{\partial W}, \quad D_0 T_{m,2} = -\{G_{0,2};\, T_{m,2}\}$$

where $T_{m,2}$ and $G^*_{m,2}$ are terms of power $m$ of the expansion of the generating function of the sought transformation $T_2$ and of the new Hamiltonian $G_2{}^*$ in series in the small parameter. Functions $G'_{m,2}$ are calculated by formulas similar to (3.7). The process of solving Eqs. (5.3) can be, however, represented in a somewhat different form.

We introduce imaginary variables $q_W$ and $p_W$ defined by formulas

$$q_W = \varepsilon \sin W, \quad p_W = \varepsilon \cos W \tag{5.4}$$

After this substitution the time $W$ does not explicitly appear in the Hamiltonian $G_2$, since $\varepsilon$ and $W$ appear in function $K_2$ in (4.3) only in the form of combinations (5.4) and $\varepsilon^2$ in frequency (4.3) can be replaced by the expression $q_W{}^2 + p_W{}^2$. The obtained Hamiltonian is of the form

$$L = L_2 + \ldots + L_m + \ldots \tag{5.5}$$

$$L_m = \check{G}_{m-2,2}\varepsilon^{m-2} = \sum_{\nu_0+\ldots+\mu_N=m} g_{\nu_0\mu_0\nu_1\mu_1\ldots\nu_N\mu_N} q_W^{\nu_0} p_W^{\mu_0} \ldots q_N^{\nu_N} p_N^{\mu_N} \tag{5.6}$$

where the superscript $\check{}$ indicates that in functions marked by it $\varepsilon$ and $W$ are eliminated by the substitution (5.4); in (5.6) $m \geqslant 3$ and function $L_2$ is determined below. Note that the effect of operators with superscripts $\check{}$ and $\hat{}$ in Sect. 4 is opposite, hence function $\check{G}_{m,2}$ can be directly obtained from functions $H_{m,2}$ in the second of formulas (4.5). For this it is only necessary to carry out in functions $H_{m,2}$ the formal substitution $q_0 \to q_W$ and $p_0 \to p_W$ and use the last three of formulas (5.2).

Using the rule of composite function differentiation the operator $\Delta_0$ can be presented in the form

$$\Delta_0 = D_0 - \left[p_W \frac{\partial}{\partial q_W} - q_W \frac{\partial}{\partial p_W}\right]$$

and the operation equation (5.3) then becomes

$$\Delta_0 S_{m-2,2} = L_m' - L_m^*$$

$$\Delta_0 S_{m-2,2} = -\{L_2;\, S_{m-2,2}\}$$

$$L_2 = \frac{1}{2}(q_W^2 + p_W^2) + \frac{1}{2}\sum_{j=1}^{N} \frac{\lambda_j}{\lambda_0}(q_j^2 + p_j^2)$$

Functions $L_m'$ are calculated by the lowest functions using formulas (3.7) where operators $D_m$ are to be replaced by operators $D_{m-2,2}$, whose action on an arbitrary function of variables $q_W$, $p_W$, $q_j$, and $p_j$ $(j = 1, \ldots, N)$ is defined by the relationships

$$D_{m-2,2}F = \{F; S_{m-2,2}\} = \sum_{j=1}^{N}\left[\frac{\partial F}{\partial q_j}\frac{\partial S_{m-2,2}}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial S_{m-2,2}}{\partial q_j}\right] \tag{5.7}$$

It is thus possible to normalize instead of the nonautonomous system with Hamiltonian $G_2$ the autonomous system (but with the number of degrees of freedom increased by one) with Hamiltonian (5.5). It is assumed that in this case the normalization procedure differs from that of normalization of an autonomous system with $N+1$ degrees of freedom only by that in calculating Poissons braces in the quantities of (3.7) and in deriving the explicit form of (3.11) differentiation is carried not with respect to all variables $q_W$, $p_W$, $q_j$, and $p_j$ $(j = 1, \ldots, N)$ but, in accordance with (5.7), only with respect to variables $q_j$ and $p_j$.

Since variables $q_j$ and $p_j$ appear in Hamiltonian (5.5) in quadratic form ( in (5.6) ( $\nu_1 + \mu_1 + \ldots + \nu_N + \mu_N = 2$ )), only resonances of the form (4.5) with $n=2$ can impede normalization.

In the absence of any such resonances Hamiltonian (5.5) can be reduced to the following normal form:

$$L^* = r_W + \sum_{j=1}^{N}\left\{\frac{\lambda_j}{\lambda_0} + \sum_{m=1}^{\infty} a_{2m,j} r_W^m\right\} r_j \tag{5.8}$$

$$q^* = \sqrt{2r}\sin\varphi, \quad p^* = \sqrt{2r}\cos\varphi$$

Function (5.8) is presented in "polar" coordinates $r_W$, $\varphi_W$, $r_j$, and $\varphi_j$ related to the new variables $q_W^*$, $p_W^*$, $q_j^*$, and $p_j^*$ by formulas of the indicated form. Parameter $a_{2m,j}$ depends on parameters U of the problem.

If the resonance relationship (5.1) is satisfied, the normal form of Hamiltonian (5.5) is

$$L^* = r_W + \sum_{j=1}^{N}\left\{\frac{\lambda_i}{\lambda_0} + \sum_{m=1}^{n_0'} a_{2m,j} r_W^m\right\} r_j +$$
$$a\sqrt{r_W^{|n_0|} r_1^{n_1} r_2^{n_2}}\sin(n_1\varphi_1 + n_2\varphi_2 - n_0\varphi_W) + O(r_W^\gamma)$$
$$\gamma = \frac{|n_0|+1}{2}, \quad n_0' = \left[\frac{1}{2}|n_0|\right]$$

where parameter $a$ also depends on parameters U and brackets indicate the taking of the integral part of a number.

When $\varepsilon$ (i.e. $q_W$ and $p_W$ ) are fairly small the transformation

$$q_W, p_W, q_j, p_j \to q_W^*, p_W^*, q_j^*, p_j^* \quad (j = 1, \ldots, N) \tag{5.9}$$

is convergent for $q_W$ and $p_W$. It is carried out using formulas similar to (3.11) in which again operators $D_{m-2,2}$ from (5.7) are to be substituted for operators $D_m$. The form of these operators clearly shows, as expected, that the imaginary variables $q_W$ and $p_W$ are not affected by such normalization.

Now let us carry out the transformation inverse of (5.4) and revert to the old independent variable $t$. If, in addition, we specify the transformation $r_0, W \to r_0^*, W^*$ by formulas

$$r_0 = r_0^* + \partial S_2 / \partial W, \quad W = W^* \tag{5.10}$$

then together with (5.9) we obtain the canonical transformation which normalizes function (5.2) with respect to all variables. In the nonresonance case the normal form of that function is (previous notation is retained for $r_0$)

$$K_2^* = \Omega_0 r_0 + \sum_{j=1}^{N} \Omega_j r_j \tag{5.11}$$

and in the resonance case it is

$$K_2^* = \Omega_0 r_0 + \sum_{j=1}^{N} \Omega_j' r_j + A\varepsilon^{|n_0|} \sqrt{r_1^{n_1} r_2^{n_2}} \sin(n_1\varphi_1 + n_2\varphi_2 - n_0 W) + O(\varepsilon^{|n_0|+1}) \tag{5.12}$$

The following notation is used in these formulas:

$$\Omega_j = \sum_{m=0}^{\infty} \Omega_j^{(m)} \varepsilon^m, \quad \Omega_j' = \sum_{m=0}^{2n_0'} \Omega_j^{(m)} \varepsilon^m, \quad A = a\lambda_0 2^{-|n_0|/2}$$

$$\Omega_j^{(0)} = \lambda_j, \quad \Omega_j^{(2m-1)} = 0, \quad \Omega_j^{(2m)} = \frac{\lambda_j}{\lambda_0} \Omega_0^{(2m)} + \sum_{k=1}^{m} 2^{-k} a_{2k,j} \Omega_0^{(2m-2k)}$$

Let us consider the case of parametric resonance. Regions of parametric resonance (instability regions) issue for small $\varepsilon$ from surfaces for which in the region of variation of the problem parameters $U$ relationship (5.1) is satisfied. According to [8] on the surfaces which bound these regions in space of parameters $U$ and $\varepsilon$ the following relationships are valid:

$$|n_1(\lambda_0\Omega_1 - \lambda_1\Omega_0) + n_2(\lambda_0\Omega_2 - \lambda_2\Omega_0)| = |\lambda_0 A| \sqrt{n_1^{n_1} n_2^{n_2}} \varepsilon^{|n_0|} \tag{5.13}$$

When the right-hand side of the last relationship is greater than the left-hand, the periodic motion is unstable, and when it is smaller, we have stability in the first approximation.

Equations (5.13) of two surfaces in the space of parameters $U$ and $\varepsilon$ may be sought in the form of series in $\varepsilon$, using series expansions of $\Omega_0$, $\Omega_1$, $\Omega_2$.

6. If parameters $U$ and $\varepsilon$ of the problem are such that the considered periodic motion is stable in linear approximation, then by normalizing the linear system using the method described in Section 5, Hamiltonian (4.4) can be reduced to the form

$$K^* = K_2^* + K_3^* + K_4^* + \ldots \tag{6.1}$$

$$K_3^* = \sum_{m=0}^{\infty} \hat{K}_{m,3}$$

$$K_4^* = B_{00} r_0^2 + \left[\frac{1}{\varepsilon^2} \sum_{m=1}^{\infty} \hat{K}_{m,2}\right] r_0 + \sum_{m=0}^{\infty} \hat{K}_{m,4}$$

where $K_2^*$ is of the form (5.11) and functions $\hat{K}_{m,i}$ $(m=0, 1, \ldots; i = 2, 3, 4)$ are of order $m$ relative to $\varepsilon$ (i.e. relative to the imaginary variables $q_W$ and $p_W$ in Sect. 5) and of order $i$ relative to $q_j$ and $p_j$ $(j = 1, \ldots N)$. These functions are readily calculated by formulas similar to (3.7). For instance,

$$K_{0,3} = H_{0,3},\ K_{1,3} = H_{1,3} + D_{1,2} H_{0,3} \tag{6.2}$$

$$K_{2,3} = H_{2,3} + D_{1,2} H_{1,3} + D_{2,2} H_{0,3},\ K_{1,2} = \overline{H}_{1,2}$$

$$K_{2,2} = \overline{H}_{2,2} + D_{1,2} \overline{H}_{1,2}, \quad K_{0,4} = H_{0,4} + \frac{1}{\varepsilon^2} H_{1,2} \left[q_W \frac{\partial}{\partial p_W} - p_W \frac{\partial}{\partial q_W}\right] S_{1,2}$$

Elucidation of the question of stability in the strict (nonlinear) meaning, requires extension of the normalization process of the Hamiltonian of perturbed motion.

Normalization of Hamiltonian (6.1) can be impeded by resonances

$$\sum_{j=1}^{N} n_j \Omega_j = n_0 \Omega_0 \quad \left(\sum_{j=1}^{N} |n_j| = n = 3, 4\right) \tag{6.3}$$

In the region of variation of parameters U and $\varepsilon$ formulas (6.3) are equations of resonance surfaces of the third and fourth order and are derived similarly to the boundaries of parametric resonance regions in Sect. 5. Parameters U that satisfy relationships (4.5) are generating parameters for such surfaces.

Let us, first, consider the values of parameters U and $\varepsilon$ which are not associated with third and fourth order resonance surfaces. In that case form $K_3^*$ in Hamiltonian (6.1) can be completely eliminated by using the method described in Sect. 5. Normalization of fourth order terms consists of the following three independent stages.

a) Normalization of terms proportional to $r_0^2$. These terms are already normalized.

b) Normalization of terms proportional to $r_0$. It can be shown that the normalization of these terms reduces to the averaging of function $\hat{K}_{1,2} + \hat{\overline{K}}_{2,2} + \ldots$ with respect to rapid phases of motion determined by Hamiltonian (5.11). Note that for $n = 2$ resonances (6.2) do not impede the normalization of these terms, since they appear only at the boundaries of parametric resonance regions and, consequently have been already taken into account in linear normalizarion.

c) Normalization of terms independent of $r_0$. This normalization stage is similar to linear normalization procedure.

As the result, Hamiltonian (6.1) of perturbed motion can in the nonresonance case be reduced to the following normal form (previous notation is used for variables):

$$K = K_2 + K_4 + K^*$$
$$K_2(r_0, r_1, \ldots, r_N) = \sum_{i=0}^{N} \Omega_i r_i, \quad K_4(r_0, r_1, \ldots, r_N) = \sum_{0 \leqslant i \leqslant j \leqslant N} B_{ij} r_i r_j \tag{6.4}$$

where the expansion of coefficients of form $K_4$ in series in $\varepsilon$ is similar to that of $B_{00}$ in (4.4), and $K^*$ is a $2\pi$-periodic function of angle variables $W, \varphi_1, \ldots, \varphi_N$; its order relative to $r_i$ is not less than $^5/_2$.

In the case of the third order resonance the normal form is

$$K = K_2 + A\varepsilon^{|n_0|} \sqrt{r_1^{|n_1|} \ldots r_N^{|n_N|}} \sin(n_1\varphi_1 + \ldots + n_N\varphi_N - n_0 W) + K^* \tag{6.5}$$

and in that of the fourth order resonance it is

$$K = K_2 + K_4 + A\varepsilon^{|n_0|} \sqrt{r_1^{|n_1|} \ldots r_N^{|n_N|}} \sin(n_1\varphi_1 + \ldots + n_N\varphi_N - n_0 W) + K^* \tag{6.6}$$

The order of function $K^*$ relative to $\varepsilon$ in (6.5) and (6.6) is not lower than $|n_0| + 1$, and the quantities $B_{ij}$ in (6.6) are determined with that accuracy.

7. Thus for determining the stability of a periodic motion it is only necessary to calculate the coefficients of one of the normal forms (6.4)-(6.6) and apply the stability criteria from [4, 9-14]. Comprehensive results may be obtained in this manner for systems with two degrees of freedom $(N = 1)$.

If third or fourth order resonances (6.3) appear in a multidimensional Hamiltonian system and in the array of numbers $n_1, \ldots, n_N$ at least one change of sign takes place, the periodic motion is formally stable [9], i.e. it is stable in any approximation.

When third order resonance (6.3) is present and in (6.5) $A \neq 0$, the periodic motion is unstable [10, 11]. If $A = 0$, the question of stability is not resolved by terms of that order.

If fourth order resonance (6.3) is present, and in the normal form (6.6)

$$|K_4(-n_0, n_1, \ldots, n_N)| < |A| \sqrt{n_1^{n_1} \ldots n_N^{n_N}} \varepsilon^{|n_0|} \tag{7.1}$$

the periodic motion is unstable [10, 11]. With the opposite sign in this inequality in the case of two-frequency Hamiltonian systems we have stability [12], and in the multidimensional case stability is shown in the last (fourth) approximation [11].

In the nonresonance case of systems with two degress of freedom the question of stability is resolved by the Arnold—Moser theorem, viz. if (in notation of (6.4) and (4.4))

$$D \neq 0 \tag{7.2}$$
$$D = K_4(\Omega_1, -\Omega_0, 0, \ldots, 0) = c_{00}\lambda_1^2 - c_{01}\lambda_0\lambda_1 + c_{11}\lambda_0^2 + O(\varepsilon^2) \tag{7.3}$$

the periodic motion is stable [4, 13].

The state of development of the theory of Hamiltonian systems does not provide means for obtaining a similarly complete result in the multidimensional case. It is only possible to make the following statement.

If for $r_0 = r_1 = \ldots = r_N = 0$ the determinants

$$D_3 = \det \left\| \frac{\partial^2 K_4}{\partial r_i \partial r_j} \right\|, \quad D_4 = \det \left\| \begin{matrix} \frac{\partial^2 K_4}{\partial r_i \partial r_j} & \frac{\partial K_2}{\partial r_i} \\ \frac{\partial K_2}{\partial r_j} & 0 \end{matrix} \right\| \tag{7.4}$$

do not simultaneously vanish, stability is present for the majority (in the meaning of the Lebesgue measure) of initial conditions [13].

It is also possible to consider the problem of formal stability of periodic motions. In the considered case the sufficient condition of formal stability reduces (see [14] and the footnote on p. 52 ) to the check of incompatibility of the system of equations (relative to $r_0, r_1, \ldots, r_N$)

$$K_2 = 0, \ K_4 = 0 \tag{7.5}$$

in the region $r_1 \geqslant 0, \ldots, r_N \geqslant 0$ (note that by definition (4.1) the sign of parameter $r_0$ is arbitrary).

In determinants (7.4) and Eqs. (7.5) it is evidently, reasonable to take into account only the principal terms of expansions of $\Omega_i$, and $B_{ij}$ in (6.4) (see also(7.3)). This means that for solving the question of stability in the nonresonance case, it is possible as a rule, to restrict the analysis to and including terms $H_4$ of the input Hamiltonian (1.1).

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